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ON WEIGHTED HARMONIC BERGMAN SPACES

Abstract. This paper is devoted to the investigation of the weighted Bergman harmonic spaces $b^p_\alpha(B)$ in the unit ball in $\mathbf{R}^n$. The reproducing kernel R_α for the ball is constructed and the integral representation for functions in $b^p_\alpha(B)$ by means of this kernel is obtained. Besides an linear mapping between the $b^2_\alpha(B)$ spaces and the ordinary L^2-space on the unit sphere, which has an explicit form of integral operator along with its inversion, is established.

Introduction

This paper is devoted to the investigation of the weighted Bergman harmonic spaces $b^p_\alpha(B)$ in the unit ball in $\mathbf{R}^n$. In Section 1 we introduce the spaces $b^p_\alpha(B)$ and prove some preliminary statements. Section 2 is devoted to the construction of reproducing kernel R_α, to the integral representation of $b^p_\omega(B)$ by means of R_α (Theorems 1 and 2) and to the orthogonal projection from $L^p(B, dV_\alpha)$ to $b^2_\alpha(B)$ (Theorem 3). Section 3 gives an integral representation of the considered spaces $b^2_\alpha(B)$ over the unit sphere. This leads to an linear mapping between the $b^2_\alpha(B)$ spaces and the ordinary L^2-space on the unit sphere, which has an explicit form of integral operator along with its inversion (Theorems 4 and 5).

1. Bergman spaces

We start by some notation which we use all over the paper.

$B = \{x \in \mathbf{R}^n : \ |x| < 1\}$ is the open unit ball in $\mathbf{R}^n$ and S is its boundary, i.e. S is the unit sphere in $\mathbf{R}^n$;

σ is the normalized surface-area measure on S, so that $\sigma(S) = 1$;

$\mathcal{H}_m(\mathbf{R}^n)$ is the set of all complex-valued homogeneous harmonic polynomials of degree m in $\mathbf{R}^n$;

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$\mathcal{H}_m(S)$ is the set of all spherical harmonics of degree m, i.e. the restrictions of functions from $\mathcal{H}_m(\mathbf{R}^n)$ on the sphere S;

$Z_m(x,y)$ is the zonal harmonic of degree m;

$P[u]$ denotes the Poisson integral of u:

$$(1)\qquad P[u](x) = \int_S P(x,\zeta)u(\zeta)\,d\sigma(\zeta), \quad \text{where} \quad P(x,\zeta) = \frac{1-|x|^2}{|\zeta - x|^n}.$$

For $1 \leqslant p < +\infty$ and $-1 < \alpha < +\infty$ the *weighted Bergman space* $b^p_\alpha(B)$ of the unit ball is the space of harmonic functions in $L^p(B, dV_\alpha)$, where

$$dV_\alpha(x) = \left(1 - |x|^2\right)^\alpha dV(x), \text{ and } dV(x) \text{ is the Lebesguae measure.}$$

When u is in $L^p(B, dV_\alpha)$, we write

$$\|u\|_{p,\alpha} = \Big[\int_B |u(x)|^p\, dV_\alpha(x)\Big]^{1/p}.$$

The next assertion states the continuity of ϱ-dilatation in $b^p_\alpha(B)$.

PROPOSITION 1. *Let $u \in b^p_\alpha(B)$ and $u_\varrho(x) = u(\varrho x)$. Then $\|u_\varrho - u\|_{p,\alpha} \to 0$ as $\varrho \to 1-0$.*

Proof. Using the expression of the volume element in polar coordinates

$$(2)\qquad dV(x) = nV(B)\, r^{n-1}\, dr\, d\sigma(\zeta)$$

(see, for instance, [3]), for any $\delta \in (0,1)$

$$(3)\qquad \|u_\varrho - u\|^p_{p,\alpha} \le \int_{|x|<\delta} |u(\varrho x) - u(x)|^p\, dV_\alpha(x)$$
$$+ 2^p nV(B)\int_\delta^1 \Big\{\int_S (|u(\varrho r\zeta)|^p + |u(r\zeta)|^p) d\sigma(\zeta)\Big\} r^{n-1}\left(1-r^2\right)^\alpha dr$$

since $(a+b)^p \le 2^p(a^p + b^p)$ $(a, b > 0)$. Further $m(\varrho) = \int_S |u(\varrho r\zeta)|^p d\sigma(\zeta)$ is nondecreasing and $m(\varrho) \le m(1)$ since $|u(x)|^p$ is subharmonic. Hence by (3)

$$\|u_\varrho - u\|^p_{p,\alpha} \le \int_{|x|<\delta} |u(\varrho x) - u(x)|^p\, dV_\alpha(x) + 2^{p+1} \int_{\delta\leqslant|x|<1} |u(x)|^p dV_\alpha(x).$$

It remains to see that the right-hand side of this inequality can be made arbitrarily small by taking δ and then ϱ close enough to 1. □

It is well known that any function harmonic in a domain containing $\overline{B}$ can be uniformly approximated on $\overline{B}$ by harmonic polynomials. Using this fact, one can prove the following corollary of Proposition 1.

COROLLARY 1. *Harmonic polynomials are dense in $b^p_\alpha(B)$.*

The following proposition shows that the point evaluation is continuous on $b^p_\alpha(B)$.

PROPOSITION 2. *For any function $u \in b^p_\alpha(B)$ and any point $x \in B$*

$$|u(x)| \le \frac{2^{n/p}}{(1-|x|)^{(n-1)/p}} \Big(nV(B) \int\limits_{(1+|x|)/2}^{1} r^{n-1} \left(1-r^2\right)^\alpha dr \Big)^{-1/p} \|u\|_{p,\alpha}.$$

Proof. The following estimates obviously are true for the Poisson's kernel (1):

$$P(x,\zeta) = \frac{1-|x|^2}{|\zeta - x|^n} \le \frac{1+|x|}{(1-|x|)^{n-1}} \le \frac{2}{(1-|x|)^{n-1}}. \tag{4}$$

Let $x \in B$ and $|x| < R < 1$. Using the subharmonicity of the function $|u(Rx)|^p$ in the neighborhood of the ball $\overline{B}$ and (4), we get

$$|u(Rx)|^p \le \int\limits_S |u(R\zeta)|^p P(x,\zeta)\, d\sigma(\zeta) \le \frac{2}{(1-|x|)^{n-1}} \int\limits_S |u(R\zeta)|^p \, d\sigma(\zeta). \tag{5}$$

Let $x = r\zeta$, where $r = |x|$, $\zeta \in S$. Using (2) and taking into account, that the integral means $M(R) = \int_S |u(R\zeta)|^p \, d\sigma(\zeta)$ is nondecreasing by R, we get

$$\begin{aligned} nV(B) \int\limits_R^1 r^{n-1} \left(1-r^2\right)^\alpha dr \int\limits_S |u(R\zeta)|^p \, d\sigma(\zeta) \le \\ \le nV(B) \int\limits_R^1 \int\limits_S |u(r\zeta)|^p \, r^{n-1} \left(1-r^2\right)^\alpha dr d\sigma(\zeta) \\ = \int\limits_{R<|x|<1} |u(x)|^p \left(1-|x|^2\right)^\alpha dV(x) \le \|u\|^p_{p,\alpha}. \end{aligned} \tag{6}$$

By (5) and (6)

$$|u(Rx)|^p \le \frac{2}{(1-|x|)^{n-1}} \Big(nV(B) \int\limits_R^1 r^{n-1} \left(1-r^2\right)^\alpha dr \Big)^{-1} \|u\|^p_{p,\alpha},$$

and the change of a variable $Rx \mapsto x$ gives

$$|u(x)| \le \frac{2^{1/p}}{(R-|x|)^{(n-1)/p}} \Big(nV(B) \int\limits_R^1 r^{n-1} \left(1-r^2\right)^\alpha dr \Big)^{-1/p} \|u\|_{p,\alpha}.$$

Taking $R = (1+|x|)/2$ we come to our assertion. □

PROPOSITION 3. *For any $1 \le p < \infty$, $b^p_\alpha(B)$ is closed subset of $L^p(B, dV_\alpha)$.*

Proof. Suppose $\|u_j - u\|_{p,\alpha} \to 0$ as $j \to \infty$, where u_j is a sequence of functions in $b^p_\alpha(B)$ and $u \in L^p(B, dV_\alpha)$. We shall show that u is equivalent to some function harmonic on B.

Let $K \in B$ be a compact. Proposition 2 implies that there exists a constant $C \equiv C(K, p, \alpha)$ such that

$$\max_{x\in K} |u(x)| \le C\|u\|_{p,\alpha}$$

for any $u \in b^p_\alpha(B)$. Hence $|u_j(x) - u_k(x)| \le C\|u_j - u_k\|_{p,\alpha}$ for any $x \in K$ and j, k. The sequence u_j is fundamental in $b^p_\alpha(B)$, and hence u_j converges uniformly on compact subsets of B to a function v harmonic on B. Besides, $u_j \to u$ in $L^p(B, dV_\alpha)$. Therefore, by Riesz' theorem there exists a subsequence of u_j converging to u pointwise almost everywhere in B. Thus, $u = v$ almost everywhere in B, and $u \in b^p_\alpha(B)$. □

COROLLARY 2. *$b^p_\alpha(B)$ is a Banach space.*

2. Reproducing kernels

Taking $p = 2$, we see that the last proposition shows that $b^2_\alpha(B)$ is a Hilbert space with inner product

$$\langle u, v\rangle = \int_B u\bar{v}\, dV_\alpha.$$

It follows from the Proposition 2, that the map $u \mapsto u(x)$ is a bounded linear functional on $b^2_\alpha(B)$ for each $x \in B$. Hence there exist a unique function $R_\alpha(x, \cdot) \in b^2_\alpha(B)$ such that $u(x) = \langle u, R_\alpha(x, \cdot)\rangle$. The reasoning similar to those in [1] shows that R_α is real valued, hence

$$u(x) = \int_B u(y) R_\alpha(x, y)\, dV_\alpha(y) \tag{7}$$

for every $u \in b^2_\alpha(B)$. The function R_α is called the *reproducing kernel* of B. For the constructing of R_α we previously prove the following

LEMMA 1. *If $m \ne k$, then $\mathcal{H}_m(\mathbf{R}^n)$ is orthogonal to $\mathcal{H}_k(\mathbf{R}^n)$ in $b^2_\alpha(B)$.*

Proof. Let $p \in \mathcal{H}_m(\mathbf{R}^n)$, $q \in \mathcal{H}_k(\mathbf{R}^n)$ and $x = r\zeta$, with $r = |x|$, $\zeta \in S$. Using formula (2) and the homogeneity of p and q,

$$\begin{aligned}\int_B p(x)\bar{q}(x)\, dV_\alpha(x) &= nV(B)\int_0^1 r^{n-1}\left(1 - r^2\right)^\alpha dr \int_S p(r\zeta)\bar{q}(r\zeta)\, d\sigma(\zeta)\\ &= nV(B)\int_0^1 r^{p+q+n-1}\left(1 - r^2\right)^\alpha dr \int_S p(\zeta)\bar{q}(\zeta)\, d\sigma(\zeta) = 0.\end{aligned}$$

The last equality follows by the orthogonality of the spherical harmonics of different degrees. □

THEOREM 1. *If $x, y \in B$ then*

$$R_\alpha(x,y) = \frac{2}{nV(B)} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)} Z_m(x,y) \tag{8}$$

The series on the right-hand side of (8) converges absolutely and uniformly on the set $\{(x,y) \in \mathbf{R}^{2n}\colon\ |x||y| \le q,\ 0<q<1\}$ and particularly on $K \times \overline{B}$, where K is arbitrary compact subset of B.

Proof. Note, that in (8) we suppose that the zonal harmonics Z_m are harmonically extended on $\mathbf{R}^n \times \mathbf{R}^n$. Let $x = r\zeta,\ y = \rho\eta$, where $\zeta,\ \eta \in S$. Taking into account that the function $Z_k(x,y)$ is homogeneous by both variables, we obtain

$$|Z_k(x,y)| = r^k\rho^k|Z_k(\zeta,\eta)| \le r^k\rho^k d_k, \tag{9}$$

where d_k is the dimension of $\mathcal{H}_k(S)$. The desired convergence follows from (9) in view of the estimate $d_k \le Ck^{n-2}$ from [1] and by virtue of Stirling's formula. Thus if $F(x,y)$ denotes the right-hand side of (8), then $F(x,\cdot)$ is a bounded harmonic function on B for each $x \in B$. In particular, $F(x,\cdot) \in b^2_\alpha(B)$ for each $x \in B$.

Now fix $x \in B$. Recall that the zonal harmonics are reproducing kernels for the space $\mathcal{H}_m(\mathbf{R}^n)$. Thus for $u \in \mathcal{H}_m(\mathbf{R}^n)$

$$u(x) = \int_S u(\zeta) Z_m(x,\zeta)\, d\sigma(\zeta) \tag{10}$$

for each $x \in \mathbf{R}^n$. We derive the analogue of (10) for integration over B with respect of measure dV_α. For every $u \in \mathcal{H}_m(\mathbf{R}^n)$ we have

$$\begin{aligned}
\int_B u(y) Z_m(x,y)\, dV_\alpha(y) &= nV(B) \int_0^1 r^{n-1}\left(1-r^2\right)^\alpha \int_S u(r\zeta) Z_m(x,r\zeta)\, d\sigma(\zeta)\, dr \\
&= nV(B) \int_0^1 r^{n+2m-1}\left(1-r^2\right)^\alpha \int_S u(\zeta) Z_m(x,\zeta)\, d\sigma(\zeta)\, dr \\
&= \frac{nV(B)}{2} u(x) \int_0^1 t^{\frac{n}{2}+m-1}(1-t)^\alpha\, dt \\
&= \frac{nV(B)}{2} \frac{\Gamma(\frac{n}{2}+m)\Gamma(\alpha+1)}{\Gamma(\frac{n}{2}+m+\alpha+1)} u(x)
\end{aligned}$$

for each $x \in \mathbf{R}^n$. Taking into account the orthogonality in $b^2_\alpha(B)$ of homogeneous harmonic polynomials of different degrees, we receive that $u(x) = \langle u, F(x,\cdot)\rangle$ whenever u is harmonic polynomial. Because point evaluation is continuous in b^2_α due to Proposition 2 and harmonic polynomials are dense in $b^2_\alpha(B)$ (see the Corollary 1), we have $u(x) = \langle u, F(x,\cdot)\rangle$ for all $u \in b^2_\alpha(B)$. Hence F is the reproducing kernel. □

It is easy to see that integral representation (7) is true not only for $b^2_\alpha(B)$, but also for any function $u \in b^p_\alpha(B)$:

Theorem 2. *Let $u \in b^p_\alpha(B)$, $1 \leqslant p < +\infty$. Then*

$$u(x) = \int\limits_B u(y) R_\alpha(x,y)\, dV_\alpha(y) \tag{11}$$

The right-hand side integral of (11) defines the orthogonal projection of $L^2(B, dV_\alpha)$ onto its subspace $b^2_\alpha(B)$, i.e. the following assertion is true.

Theorem 3. *The operator*

$$Q_\alpha[u](x) = \int\limits_B u(y) R_\alpha(x,y)\, dV_\alpha(y), \quad u \in L^2(B, dV_\alpha), \quad x \in B,$$

is the orthogonal projection of $L^2(B, dV_\alpha)$ onto $b^2_\alpha(B)$.

Proof. As $L^2(B, dV_\alpha) = b^2_\alpha(B) \oplus \big(b^2_\alpha(B)\big)^\perp$, any $u \in L^2(B, dV_\alpha)$ can be written in the form $u = u_1 + u_2$, where $u_1 \in b^2_\alpha(B)$ and $u_2 \in \big(b^2_\alpha(B)\big)^\perp$. Hence $Q_\alpha[u] = Q_\alpha[u_1] + Q_\alpha[u_2]$, where $Q_\alpha[u_1] = u_1$ by Theorem 2. On the other hand,

$$Q_\alpha[u_2](x) = \int\limits_B u_2(y) R_\alpha(x,y)\, dV_\alpha(y) = \langle u_2, R_\alpha(x,\cdot)\rangle_\alpha = 0,$$

since due to Theorem 1 for a fixed $x \in B$ the function $R_\alpha(x,y)$ is harmonic in y on a domain containing $\overline{B}$, and u_2 is orthogonal to $b^2_\alpha(B)$. Thus $Q_\alpha[u] = u_1$, i.e. Q_α is the orthogonal projector $L^2(B, dV_\alpha) \mapsto b^2_\alpha(B)$. □

We suppose that for any $x \in B$ the Poisson kernel $P(x,y)$ can be harmonically extended to $\overline{B}$ as follows:

$$P(x,y) = \frac{1 - |x|^2|y|^2}{(1 - 2x\cdot y + |x|^2|y|^2)^{\frac{n}{2}}},$$

where $\cdot$ denotes the usual Euclidean inner product. To obtain an expression of R_α by means of the Poisson kernel P, we use some well known facts from the theory of fractional integro-differentiation in the Riemann-Liouville sense. The primitive of $f \in L^1(0,1)$ of order $\alpha > 0$ is defined as

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau.$$

The derivative of order α is defined to be

$$D^\alpha f(t) = \frac{d^p}{dt^p}\left\{D^{-(p-\alpha)} f(t)\right\},$$

where the integer p is determined by the inequalities $p-1<\alpha\le p$. Using the simple equality

$$D^{\alpha+1}t^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(\gamma-\alpha)}t^{\gamma-\alpha-1},$$

we find that

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{\Gamma(\frac{n}{2}+m+\alpha+1)}{\Gamma(\frac{n}{2}+m)}Z_m(x,y)&=\sum_{m=0}^{\infty}D^{\alpha+1}\left(t^{\frac{n}{2}+m+\alpha}Z_m(x,y)\right)\Big|_{t=1}\\ =D^{\alpha+1}\Big(\sum_{m=0}^{\infty}t^{\frac{n}{2}+m+\alpha}Z_m(x,y)\Big)\Big|_{t=1}&=D^{\alpha+1}\Big(\sum_{m=0}^{\infty}t^{\frac{n}{2}+\alpha}Z_m(tx,y)\Big)\Big|_{t=1}\\ &=D^{\alpha+1}\Big(t^{\frac{n}{2}+\alpha}P(tx,y)\Big)\Big|_{t=1}.\end{aligned}$$

Thus

$$R_\alpha(x,y)=\frac{2}{n\Gamma(\alpha+1)V(B)}D^{\alpha+1}\left(t^{\frac{n}{2}+\alpha}P(tx,y)\right)\Big|_{t=1}.$$

When α is a nonnegative integer, the operator $D^{\alpha+1}$ is the usual derivation, and this allows to calculate $R_\alpha(x,y)$ in an explicit form. Particularly, for $\alpha=0$ this calculation results in the formula

$$R_0(x,y)=\frac{2}{nV(B)}\,\frac{d}{dt}\left(t^{\frac{n}{2}}P(tx,y)\right)\Big|_{t=1}=\frac{nP(x,y)+2\frac{d}{dt}\,P(tx,y)|_{t=1}}{nV(B)},$$

which coincides with that of [1] in view of

$$2\frac{d}{dt}P(tx,y)\Big|_{t=1}=\frac{d}{dt}P(tx,ty)\Big|_{t=1}.$$

3. Representation of $b^2_\alpha(B)$ over the sphere

For $1\leqslant p\leqslant\infty$ denote by $h^p(B)$ the harmonic Hardy space, i.e. the class of functions u harmonic on B for which

$$\|u\|_{h^p}=\sup_{0\leqslant r<1}\|u_r\|_{L^p(S)}<\infty.$$

Proposition 4. *Let f be a harmonic function in B and let*

$$f(x)=\sum_{k=0}^{\infty}p_k(x)$$

be the homogeneous expansion of f in B. Then $f\in h^2(B)$ if and only if

$$\sum_{k=0}^{\infty}\|p_k\|^2_{L^2(S)}<\infty.$$

P r o o f. For any $r \in (0,1)$

$$\|f_r\|^2_{L^2(S)} = \int_S |f_r(\zeta)|^2\, d\sigma(\zeta) = \int_S \Big(\sum_{k=0}^{\infty} p_k(r\zeta)\Big)\Big(\sum_{j=0}^{\infty} \overline{p}_j(r\zeta)\Big) d\sigma(\zeta)$$
$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \int_S p_k(r\zeta)\overline{p}_j(r\zeta)\, d\sigma(\zeta) = \sum_{k=0}^{\infty} \int_S |p_k(r\zeta)|^2\, d\sigma(\zeta),$$

and the passage $r \to 1$ gives

$$\|f\|^2_{h^2} = \sum_{k=0}^{\infty} \|p_k\|^2_{L^2(S)} : \qquad \square$$

PROPOSITION 5. *Let u be a harmonic function in B and $u(x) = \sum_{k=0}^{\infty} u_k(x)$ be its homogeneous expansion. Then $u \in b^2_\alpha(B)$ if and only if*

$$\sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2}+\alpha+1+k)} \|u_k\|^2_{L^2(S)} < +\infty. \tag{12}$$

P r o o f. For any $r \in (0,1)$

$$\int_{B(r)} |u(y)|^2\, dV_\alpha(y) = \int_{B(r)} |u(y)|^2 \left(1-|y|^2\right)^\alpha dV(y)$$
$$= nV(B)\int_0^r \rho^{n-1}\left(1-\rho^2\right)^\alpha d\rho\Big(\sum_{k=0}^{\infty} u_k(\rho\zeta)\sum_{s=0}^{\infty} \overline{u}_s(\rho\zeta)\Big)\, d\sigma(\zeta)$$
$$= nV(B)\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\int_0^r \rho^{n-1+k+s}\left(1-\rho^2\right)^\alpha d\rho \int_S u_k(\zeta)\overline{p}_s(\zeta)\, d\sigma(\zeta)$$
$$= \frac{nV(B)}{2}\sum_{k=0}^{\infty}\int_0^{r^2} t^{\frac{n}{2}-1+k}(1-t)^\alpha\, dt \int_S |u_k(\zeta)|^2\, d\sigma(\zeta).$$

Taking into account that

$$\int_0^1 t^{\frac{n}{2}-1+k}(1-t)^\alpha\, dt = \frac{\Gamma(\frac{n}{2}+k)\Gamma(\alpha+1)}{\Gamma(\frac{n}{2}+\alpha+1+k)}$$

and letting $r \to 1-0$ we get

$$\|u\|^2_{2,\alpha} = \int_B |u(y)|^2\, dV_\alpha(y) = \frac{nV(B)\Gamma(\alpha+1)}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2}+\alpha+1+k)} \|u_k\|^2_{L^2(S)}.$$

$\square$

Reminding that $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$, we prove

PROPOSITION 6. *Let $f \in L^2(S)$ and let $f = \sum_{m=0}^{\infty} p_m$ be its spherical harmonic expansion (i.e. $p_m \in \mathcal{H}_m(S)$ and the sum converges in $L^2(S)$).*

Then the following formulas are true for homogeneous harmonic polynomials $p_m(x)$*:*

$$p_m(x) = \int_S f(\zeta) Z_m(x,\zeta)\, d\sigma(\zeta), \quad m = 0, 1, \ldots \tag{13}$$

P r o o f. For any fixed $x = r\eta$ $(r \geq 0,\ \eta \in S)$

$$\begin{aligned} p_m(x) = r^m p_m(\eta) &= r^m \int_S p_m(\zeta) Z_m(\eta,\zeta)\, d\sigma(\zeta) \\ &= r^m \int_S \Big(\sum_{k=0}^{\infty} p_k(\zeta)\Big) Z_m(\eta,\zeta)\, d\sigma(\zeta) \\ &= r^m \int_S f(\zeta) Z_m(\eta,\zeta)\, d\sigma(\zeta) \\ &= \int_S f(\zeta) Z_m(x,\zeta)\, d\sigma(\zeta). \end{aligned}$$

where the third equality follows by the orthogonality of the spherical harmonics of different degrees. □

THEOREM 4. *Let* $u \in b^2_\alpha(B)$ *and*

$$f(x) = \int_0^1 u(tx) t^{\frac{n}{2}-1} (1-t)^{\frac{\alpha-1}{2}}\ dt.$$

Then $f \in h^2(B)$ *and*

$$u(x) = \frac{nV(B)}{2} \int_S f(\zeta) R_{\frac{\alpha-1}{2}}(x,\zeta)\, d\sigma(\zeta).$$

P r o o f. Let $u(x) = \sum_{k=0}^{\infty} u_k(x)$ be the homogeneous expansion. Then

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} u_k(x) \int_0^1 t^{\frac{n}{2}-1+k} (1-t)^{\frac{\alpha-1}{2}}\ dt \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2}+k)\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{n}{2}+\frac{\alpha+1}{2}+k)} u_k(x) = \sum_{k=0}^{\infty} p_k(x), \end{aligned}$$

where

$$p_k(x) = \frac{\Gamma(\frac{n}{2}+k)\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{n}{2}+\frac{\alpha+1}{2}+k)} u_k(x). \tag{14}$$

By Stirling's formula

$$\lim_{k\to\infty} \frac{\Gamma(\frac{n}{2}+k)}{\Gamma(\frac{n}{2}+\alpha+1+k)} \cdot \frac{\Gamma^2(\frac{n}{2}+\frac{\alpha+1}{2}+k)}{\Gamma^2(\frac{n}{2}+k)} = 1. \tag{15}$$

¿From convergence of series in (12) and from (15) it follows, that

$$\sum_{k=0}^{\infty} \|p_k\|^2_{L^2(S)} = \sum_{k=0}^{\infty} \frac{\Gamma^2(\frac{n}{2}+k)\Gamma^2(\frac{\alpha+1}{2})}{\Gamma^2(\frac{n}{2}+\frac{\alpha+1}{2}+k)} \|u_k\|^2_{L^2(S)} < +\infty,$$

and due to Proposition 4 this means, that $f \in h^2(B)$. Further, using (13), (14) and (8), we obtain

$$\begin{aligned} u(x) &= \sum_{k=0}^{\infty} u_k(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2}+\frac{\alpha+1}{2}+k)}{\Gamma(\frac{n}{2}+k)\Gamma(\frac{\alpha+1}{2})} p_k(x) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2}+\frac{\alpha+1}{2}+k)}{\Gamma(\frac{n}{2}+k)\Gamma(\frac{\alpha+1}{2})} \int_S f(\zeta) Z_k(x,\zeta)\, d\sigma(\zeta) \\ &= \int_S f(\zeta) \Big[\sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2}+\frac{\alpha+1}{2}+k)}{\Gamma(\frac{n}{2}+k)\Gamma(\frac{\alpha+1}{2})} Z_k(x,\zeta) \Big] d\sigma(\zeta) \\ &= \frac{nV(B)}{2} \int_S f(\zeta) R_{\frac{\alpha-1}{2}}(x,\zeta)\, d\sigma(\zeta). \qquad \square \end{aligned}$$

Theorem 5. *The operator*

$$T_\alpha[f](x) = \frac{nV(B)}{2} \int_S f(\zeta) R_{\frac{\alpha-1}{2}}(x,\zeta)\, d\sigma(\zeta)$$

maps one-to-onely $L^2(S)$ onto $b^2_\alpha(B)$; in other words, the formula $u(x) = T_\alpha[f](x)$ is a parametric representation[1] *of $b^2_\alpha(B)$.*

Proof. Let f is an arbitrary function in $L^2(S)$, and let $f = \sum p_k$ be its decomposition ($p_k \in \mathcal{H}_k(S)$). Then the function

$$P[f](x) = \sum_{k=0}^{\infty} p_k(x),$$

where $P[f]$ is a Poisson integral of f, belongs to $h^2(B)$. The function

$$u(x) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{2}+\frac{\alpha+1}{2}+k)}{\Gamma(\frac{n}{2}+k)\Gamma(\frac{\alpha+1}{2})} p_k(x)$$

belongs to $b^2_\alpha(B)$, due to (15) and Propositions 4, 5. Using the Proposition 6 as above we obtain

$$u(x) = \frac{nV(B)}{2} \int_S f(\zeta) R_{\frac{\alpha-1}{2}}(x,\zeta)\, d\sigma(\zeta). \qquad \square$$

[1]The term *parametric representation* is used for a representation, which completely describes the considered class of functions, i.e. any function of the class is representable in some form and any function representable in that form belongs to the considered class.

Note that at first the parametric representation for the weighted classes of functions, analytic on the unit disk in the complex plain, is given by M. M. Djrbashian in [2]:

References

[1] Sh. Axler, P. Bourdon, W. Ramsey, *Harmonic Function Theory*, Springer-Verlag New York, Inc., 2001.
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[3] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987.

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